

A NOTE ON TRACES OF SINGULAR MODULI

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ABSTRACT. We will generalize Osburn's work ([6]) about a congruence for traces defined in terms of Hauptmodul associated to certain genus zero groups of higher levels.

1. INTRODUCTION

Let \mathfrak{H} denote the complex upper half-plane and $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. For an integer $N \geq 2$ let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e|N$. There are only finitely many N for which the modular curve $\Gamma_0(N)^* \backslash \mathfrak{H}^*$ has genus 0 ([5]). In particular, if we let \mathfrak{S} be the set of such N which are prime, then

$$\mathfrak{S} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For each $p \in \mathfrak{S}$ let $j_p^*(\tau)$ be the corresponding Hauptmodul with a Fourier expansion of the form $q^{-1} + O(q)$ where $q := e^{2\pi i \tau}$.

Let $p \in \mathfrak{S}$. For an integer $d \geq 1$ such that $-d \equiv \square \pmod{4p}$ let \mathcal{Q}_d be the set of all positive definite integral binary quadratic forms $Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$ of discriminant $-d = b^2 - 4ac$. To each $Q \in \mathcal{Q}_d$ we associate the unique root $\alpha_Q \in \mathfrak{H}$ of $Q(x, 1)$. Consider the set

$$\mathcal{Q}_{d,p} := \{[a, b, c] \in \mathcal{Q}_d : a \equiv 0 \pmod{p}\}$$

on which $\Gamma_0(p)^*$ acts. We then define the *trace* $t^{(p)}(d)$ by

$$t^{(p)}(d) := \sum_{Q \in \mathcal{Q}_{d,p}/\Gamma_0(p)^*} \frac{1}{\omega_Q} j_p^*(\alpha_Q) \in \mathbb{Z}$$

where ω_Q is the number of stabilizers of Q in the transformation group $\pm\Gamma_0(p)^*/\pm 1$ ([4]).

Osburn ([6]) showed the following congruence:

Theorem 1.1. *Let $p \in \mathfrak{S}$. If $d \geq 1$ is an integer such that $-d \equiv \square \pmod{4p}$ and $\ell \neq p$ is an odd prime which splits in $\mathbb{Q}(\sqrt{-d})$, then*

$$t^{(p)}(\ell^2 d) \equiv 0 \pmod{\ell}.$$

Although this result is true, we think that his proof seems to be unclear. Precisely speaking, let $D \geq 1$ be an integer such that $D \equiv \square \pmod{4p}$. In §3 we shall define

$$\begin{aligned} A_\ell(D, d) &:= \text{the coefficient of } q^D \text{ in } f_{d,p}(\tau)|T_{1/2,p}(\ell^2) \\ B_\ell(D, d) &:= \text{the coefficient of } q^d \text{ in } g_{D,p}(\tau)|T_{3/2,p}(\ell^2) \end{aligned}$$

where $f_{d,p}(\tau)$ and $g_{D,p}(\tau)$ are certain half integral weight modular forms, and $T_{1/2,p}(\ell^2)$ and $T_{3/2,p}(\ell^2)$ are Hecke operators of weight 1/2 and 3/2, respectively. The key step that is not presented in Osburn's work is the fact $A_\ell(1, d) = -B_\ell(1, d)$ which would be nontrivial at all. In this paper we shall first give a proof of more general statement $A_\ell(D, d) = -B_\ell(D, d)$ (Proposition 3.1), and then further generalize Theorem 1.1 as follows,

$$t^{(p)}(\ell^{2n} d) \equiv 0 \pmod{\ell^n}$$

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for all $n \geq 1$ (Theorem 3.3).

2. PRELIMINARIES

Let k and $N \geq 1$ be integers. If $f(\tau)$ is a function on \mathfrak{H} and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, then we define the slash operator $[\gamma]_{k+1/2}$ on $f(\tau)$ by

$$f(\tau)|[\gamma]_{k+1/2} := j(\gamma, \tau)^{-2k-1} f(\gamma\tau)$$

where

$$j(\gamma, \tau) := \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{c\tau + d} \quad \text{with } \varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4} \end{cases}.$$

Here $(\frac{c}{d})$ is the Kronecker symbol and $\sqrt{c\tau + d}$ takes its argument on the interval $(-\pi/2, \pi/2]$.

We denote by $M_{k+1/2}^{+\dots+}(N)^\dagger$ the infinite dimensional vector space of weakly holomorphic modular forms of weight $k + 1/2$ on $\Gamma_0(4N)$ which satisfy the Kohnen plus condition. Namely, the space consists of the functions $f(\tau)$ on \mathfrak{H} such that

- (i) $f(\tau)$ is holomorphic on \mathfrak{H} and meromorphic at the cusps;
- (ii) $f(\tau)$ is invariant under the action of $[\gamma]_{k+1/2}$ for all $\gamma \in \Gamma_0(4N)$;
- (iii) $f(\tau)$ has a Fourier expansion of the form

$$\sum_{(-1)^k n \equiv \square \pmod{4N}} a(n) q^n.$$

Suppose that ℓ is a prime with $\ell \nmid N$. The action of the Hecke operator $T_{k+1/2, N}(\ell^2)$ on a form

$$f(\tau) = \sum_{(-1)^k n \equiv \square \pmod{4N}} a(n) q^n \quad \text{in } M_{k+1/2}^{+\dots+}(N)^\dagger$$

is given by

$$f(\tau)|T_{k+1/2, N}(\ell^2) := \ell_k \sum_{(-1)^k n \equiv \square \pmod{4N}} \left(a(\ell^2 n) + \left(\frac{(-1)^k n}{\ell} \right) \ell^{k-1} a(n) + \ell^{2k-1} a(n/\ell^2) \right) q^n \quad (2.1)$$

where

$$\ell_k := \begin{cases} \ell^{1-2k} & \text{if } k \leq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Here $a(n/\ell^2) := 0$ if $\ell^2 \nmid n$. As is well-known, $f(\tau)|T_{k+1/2, N}(\ell^2)$ belongs to $M_{k+1/2}^{+\dots+}(N)^\dagger$.

Proposition 2.1. *Let $p \in \mathfrak{S}$.*

- (i) *For every integer $D \geq 1$ such that $D \equiv \square \pmod{4p}$ there is a unique $g_{D,p}$ in $M_{3/2}^{+\dots+}(p)^\dagger$ with the Fourier expansion*

$$g_{D,p}(\tau) = q^{-D} + \sum_{d \geq 0, -d \equiv \square \pmod{4p}} B(D, d) q^d \quad (B(D, d) \in \mathbb{Z}).$$

- (ii) *For every integer $d \geq 0$ such that $-d \equiv \square \pmod{4p}$ there is a unique form*

$$f_{d,p}(\tau) = \sum_{D \in \mathbb{Z}} A(D, d) q^D \quad (A(D, d) \in \mathbb{Z})$$

in $M_{1/2}^{+\dots+}(p)^\dagger$ with a Fourier expansion of the form $q^{-d} + O(q)$. They form a basis of $M_{1/2}^{+\dots+}(p)^\dagger$.

- (iii) *For every integer $d \geq 0$ such that $-d \equiv \square \pmod{4p}$ and every integer $D \geq 1$ such that $D \equiv \square \pmod{4p}$ we have*

$$A(D, d) = -B(D, d).$$

- (iv) *For every integer $d \geq 1$ such that $-d \equiv \square \pmod{4p}$ we get*

$$t^{(p)}(d) = -B(1, d).$$

Proof. See [1] Theorem 5.6, [3] §2.2, [4] Lemma 3.4 and Corollary 3.5. \square

3. GENERALIZATION OF THEOREM 1.1

We first prove the following necessary proposition by adopting Zagier's argument ([7] Theorem 5).

Proposition 3.1. *Let $p \in \mathfrak{S}$ and $\ell \neq p$ be a prime. For each integer $d \geq 0$ such that $-d \equiv \square \pmod{4p}$, define integers $A_\ell(D, d)$ and $B_\ell(D, d)$ in the following manner:*

$$\begin{aligned} A_\ell(D, d) &:= \text{the coefficient of } q^D \text{ in } f_{d,p}(\tau)|_{T_{1/2,p}}(\ell^2) \text{ for each integer } D \\ B_\ell(D, d) &:= \text{the coefficient of } q^d \text{ in } g_{D,p}(\tau)|_{T_{3/2,p}}(\ell^2) \text{ for each integer } D \geq 1 \\ &\quad \text{such that } D \equiv \square \pmod{4p}. \end{aligned}$$

Then we have the relation

$$A_\ell(D, d) = -B_\ell(D, d) \text{ for every integer } D \geq 1 \text{ such that } D \equiv \square \pmod{4p}.$$

Proof. For a pair of rational numbers a and b , let

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Let $d \geq 0$ be a fixed integer such that $-d \equiv \square \pmod{4p}$. It follows from the defining property of $f_{d,p}(\tau)$, namely

$$A(D, d) = \delta_{D, -d} \quad \text{if } D \leq 0$$

that if $D \leq 0$, then

$$\begin{aligned} A_\ell(D, d) &= \ell A(\ell^2 D, d) + \left(\frac{D}{\ell}\right) A(D, d) + A(D/\ell^2, d) \text{ by the definition (2.1)} \\ &= \ell \delta_{\ell^2 D, -d} + \left(\frac{D}{\ell}\right) \delta_{D, -d} + \delta_{D/\ell^2, -d} \\ &= \ell \delta_{D, -d/\ell^2} + \left(\frac{D}{\ell}\right) \delta_{D, -d} + \delta_{D, -d\ell^2}. \end{aligned}$$

Hence the principal part of $f_{d,p}(\tau)|_{T_{1/2,p}}(\ell^2)$ at infinity is

$$\ell q^{-d/\ell^2} + \left(\frac{-d}{\ell}\right) q^{-d} + q^{-d\ell^2}$$

where the first term should be omitted unless $-d/\ell^2$ is an integer. Therefore we achieve

$$f_{d,p}(\tau)|_{T_{1/2,p}}(\ell^2) = \ell f_{d/\ell^2,p}(\tau) + \left(\frac{-d}{\ell}\right) f_{d,p}(\tau) + f_{d\ell^2,p}(\tau) \text{ by Proposition 2.1(ii).} \quad (3.1)$$

And, for every integer $D \geq 1$ such that $D \equiv \square \pmod{4p}$ we derive that

$$\begin{aligned} A_\ell(D, d) &= \ell A(D, d/\ell^2) + \left(\frac{-d}{\ell}\right) A(D, d) + A(D, d\ell^2) \text{ by (3.1)} \\ &= -\ell B(D, d/\ell^2) - \left(\frac{-d}{\ell}\right) B(D, d) - B(D, d\ell^2) \text{ by Proposition 2.1(iii)} \\ &= -B_\ell(D, d) \text{ by the definition (2.1).} \end{aligned}$$

\square

On the other hand, we apply Jenkins' idea ([2]) to develop a formula for the coefficient $B(D, \ell^{2n}d)$.

Proposition 3.2. *Let $p \in \mathfrak{S}$ and $\ell \neq p$ be a prime. If $d \geq 0$ and $D \geq 1$ are integers such that $-d \equiv \square \pmod{4p}$ and $D \equiv \square \pmod{4p}$, then*

$$\begin{aligned} B(D, \ell^{2n}d) &= \ell^n B(\ell^{2n}D, d) + \sum_{t=0}^{n-1} \left(\frac{D}{\ell}\right)^{n-t-1} (B(D/\ell^2, \ell^{2t}d) - \ell^{t+1}B(\ell^{2t}D, d/\ell^2)) \\ &\quad + \sum_{t=0}^{n-1} \left(\frac{D}{\ell}\right)^{n-t-1} \left(\left(\left(\frac{D}{\ell}\right) - \left(\frac{-d}{\ell}\right) \right) \ell^t B(\ell^{2t}D, d) \right) \end{aligned}$$

for all $n \geq 1$.

Proof. From the definition (2.1) we have

$$A_\ell(D, d) = \ell A(\ell^2 D, d) + \left(\frac{D}{\ell}\right) A(D, d) + A(D/\ell^2, d) \quad (3.2)$$

$$B_\ell(D, d) = \ell B(D, d/\ell^2) + \left(\frac{-d}{\ell}\right) B(D, d) + B(D, d\ell^2). \quad (3.3)$$

Combining Proposition 3.1 with (3.2) we get

$$B_\ell(D, d) = \ell B(\ell^2 D, d) + \left(\frac{D}{\ell}\right) B(D, d) + B(D/\ell^2, d). \quad (3.4)$$

We then derive from (3.3) and (3.4) that

$$B(D, \ell^2 d) = \ell B(\ell^2 D, d) + \left(\frac{D}{\ell}\right) B(D, d) + B(D/\ell^2, d) - \ell B(D, d/\ell^2) - \left(\frac{-d}{\ell}\right) B(D, d). \quad (3.5)$$

The remaining part of the proof is exactly the same as that of [2] Theorem 1.1. Indeed, one can readily prove the proposition by using induction on n and applying only (3.5). \square

Now, we are ready to prove our main theorem which would be a generalization of Osburn's result.

Theorem 3.3. *With the same notations as in Theorem 1.1 we have*

$$t^{(p)}(\ell^{2n}d) \equiv 0 \pmod{\ell^n}$$

for all $n \geq 1$.

Proof. We achieve that

$$\begin{aligned} t^{(p)}(\ell^{2n}d) &= -B(1, \ell^{2n}d) \text{ by Proposition 2.1(iv)} \\ &= -\ell^n B(\ell^{2n}, d) - \sum_{t=0}^{n-1} \left(\frac{1}{\ell}\right)^{n-t-1} (B(1/\ell^2, \ell^{2t}d) - \ell^{t+1}B(\ell^{2t}, d/\ell^2)) \\ &\quad - \sum_{t=0}^{n-1} \left(\frac{1}{\ell}\right)^{n-t-1} \left(\left(\left(\frac{1}{\ell}\right) - \left(\frac{-d}{\ell}\right) \right) \ell^t B(\ell^{2t}, d) \right) \text{ by Proposition 3.2} \\ &= -\ell^n B(\ell^{2n}, d) \text{ by the facts that } 1/\ell^2 \text{ and } d/\ell^2 \text{ are not integers, and } \left(\frac{-d}{\ell}\right) = 1 \\ &\equiv 0 \pmod{\ell^n} \end{aligned}$$

as desired. \square

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